

THE STOKES AND POISSON PROBLEM IN VARIABLE EXPONENT SPACES

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ABSTRACT. We study the Stokes and Poisson problem in the context of variable exponent spaces. We prove existence of strong and weak solutions for bounded domains with $C^{1,1}$ boundary with inhomogeneous boundary values. The result is based on generalizations of the classical theories of Calderón-Zygmund and Agmon-Douglis-Nirenberg to variable exponent spaces.

1. INTRODUCTION

In the last decades, the generalized Lebesgue spaces $L^{p(\cdot)}$ and the corresponding generalized Sobolev spaces $W^{k,p(\cdot)}$ have attracted more and more attention. Before 1990 pioneering work has been done by Orlicz, Nakano, Hudzik, Musielak, and other authors. One of the first who studied problems with variable exponents in the context of variational integrals was Zhikov in his pioneering paper [Zhi86] and subsequent works including [Zhi95], [Zhi04], [Zhi08]. In the last twenty years, many new works have been devoted to the study of variable exponent spaces. We refer to Kováčik, Rákosník [KR91], Samko [Sam98], [Sam99], Fan, Zhao [FZ01], Cruz-Uribe, Fiorenza, Martell, Pérez [CUFMP06], Diening [Die04a], [Die04b], Diening, Růžička [DR03a], Diening, Harjulehto, Hästö, Růžička [DHHR10] for properties of these spaces such as reflexivity, denseness of smooth functions, and Sobolev type embeddings, and for the treatment of operators of harmonic analysis in the variable exponent context. The study of these spaces has been stimulated by problems in elasticity, fluid dynamics, calculus of variations, and differential equations with $p(x)$ -growth conditions. For example, in Růžička, Rajagopal [RR96] one can find a model of electrorheological fluids, where the essential part of the energy is given by $\int |Df(x)|^{p(x)} dx$, where $Df(x)$ is the symmetric part of the gradient ∇f . The same type of energy also appears in a model proposed by Zhikov [Zhi08] for the thermistor problem. This energy also appears in the investigations of variational integrals with non-standard growth, see e.g. Zhikov [Zhi86], Marcellini [Mar91], Acerbi, Mingione [AM01].

Regularity results for the Stokes system and the Poisson equation belong to the most classical problems treated in the theory of partial differential equations and fluid dynamics and often occur as auxiliary problems in the treatment of nonlinear equations. In this paper we generalize some of these results to the variable exponent context. Besides being of interest in their own as generalizations of classical results to interesting new function spaces, these results are of great importance in the analysis of the nonlinear equations occurring in the study of the fluid mechanical problems mentioned above. Of course, the whole treatment applies to a much larger class of elliptic problems.

We develop the analysis of the Stokes system in depth, while the results on the Poisson equation will be stated without proofs. For a sketch of the proofs we refer the reader to [DHHR10], for full details on both problems see [Len08]. In fact, the treatment of the Poisson equation is much simpler than that of the Stokes system

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and general elliptic problems. This is due to a symmetry of the fundamental solution of the Laplacian in the half-space by which the regularity near the boundary may be established without the use of the Agmon-Douglis-Nirenberg theory. For the Stokes system and general elliptic problems this symmetry is not granted and the full theory is needed.

The paper is organized as follows. We begin with a brief summary of elementary properties of generalized Lebesgue and Sobolev spaces which we will need in the sequel, and we introduce the concept of homogeneous Sobolev spaces in the variable exponent context, cf. [DHHR10]. Then we state the generalizations of the classical Calderón-Zygmund and Agmon-Douglis-Nirenberg theorems for symmetric kernels to generalized Lebesgue spaces. These generalizations have been treated for the first time in Růžička, Diening [DR03b], [DR03c] in a somewhat weaker form. Unfortunately, the requirements on the kernel in [DR03b], [DR03c] seem too restrictive for an application to the Stokes problem. With the help of the results of Cruz-Uribe et al [CUFMP06] on singular integrals with rough kernels the requirements can be relaxed sufficiently, cf. [DHHR10]. In the subsequent section we prove the existence and uniqueness of a strong solution in $W^{2,p(\cdot)} \times W^{1,p(\cdot)}$ of the Stokes problem in bounded domains with $C^{1,1}$ -boundary, provided that the right-hand sides are in $L^{p(\cdot)} \times W^{1,p(\cdot)}$ and the boundary values are in $\text{tr}(W^{2,p(\cdot)})$. Furthermore, we show an analogous result for weak solutions in $W^{1,p(\cdot)} \times L^{p(\cdot)}$ for the right-hand sides in $W^{-1,p(\cdot)} \times L^{p(\cdot)}$ and boundary values in $\text{tr}(W^{1,p(\cdot)})$. The main idea of the proof is a localization technique to reduce the interior and the boundary regularity to regularity results on the whole-space and the half-space, respectively. In the final section we state the analogous results for the Poisson problem, omitting the proofs.

2. VARIABLE EXPONENT SPACES

Let us introduce the variable exponent spaces $L^{p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$. Most of the following fundamental properties of these spaces can be found in [KR91], [FZ01]. We also refer to the extensive book [DHHR10] on variable exponent spaces. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. A measurable function $p : \Omega \rightarrow [1, \infty)$ is called *exponent*. If $p^+ := \sup p < \infty$, then p is called *bounded exponent*. For a bounded exponent p we define $L^{p(\cdot)}(\Omega)$ to consist of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that the *modular*

$$\rho_p(f) := \int_{\Omega} |f(x)|^{p(x)} dx$$

is finite. The expression

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : \rho_p(f/\lambda) \leq 1\}$$

defines a norm on $L^{p(\cdot)}(\Omega)$. This makes $L^{p(\cdot)}(\Omega)$ a Banach space. Moreover, one can show that $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and that $L^{p(\cdot)}(\Omega)$ is separable. Further, let $W^{k,p(\cdot)}(\Omega)$ denote the space of measurable functions $f : \Omega \rightarrow \mathbb{R}$ such that f and the distributional derivatives $f, \nabla f, \dots, \nabla^k f$ are in $L^{p(\cdot)}$. The norm $\|f\|_{k,p(\cdot)} := \sum_{i=0}^k \|\nabla^i f\|_{p(\cdot)}$ makes $W^{k,p(\cdot)}(\Omega)$ a Banach space. By $W_0^{k,p(\cdot)}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. The space $W^{-k,p(\cdot)}(\Omega)$ is defined as the dual of the space $W_0^{k,p'(\cdot)}(\Omega)$. As usual we set $1/p + 1/p' = 1$. If $p^- := \inf p > 1$, then $W^{k,p(\cdot)}(\Omega)$ is reflexive. For bounded domains $\Omega \subset \mathbb{R}^n$ with Lipschitz-continuous boundary, we define the *trace space* $\text{tr}(W^{k,p(\cdot)}(\Omega))$ by

$$\text{tr}(W^{k,p(\cdot)}(\Omega)) := \{f \in L^1(\partial\Omega) \mid \exists u \in W^{k,p(\cdot)}(\Omega) : u|_{\partial\Omega} = f\}.$$

Then

$$\|f\|_{\text{tr}(W^{k,p(\cdot)}(\Omega))} := \inf_{\substack{u \in W^{k,p(\cdot)}(\Omega), \\ u|_{\partial\Omega} = f}} \|u\|_{W^{k,p(\cdot)}(\Omega)}$$

defines a norm on $\text{tr}(W^{k,p(\cdot)}(\Omega))$ which makes the trace space a Banach space.

We have to impose some (weak) conditions on the exponent to recover important results from the classical Lebesgue and Sobolev spaces. The crucial condition is the so-called *log-Hölder* continuity of the exponent p , i.e.,

$$|p(x) - p(y)| \leq \frac{C}{\ln(e + |x - y|^{-1})}$$

for all $x, y \in \Omega$. If Ω is unbounded, then this local continuity is supplemented by the condition that there exists the limit $p(\infty) := \lim_{x \rightarrow \infty} p(x)$ and

$$|p(x) - p(\infty)| \leq \frac{C}{\ln(e + |x|)}.$$

Let us denote by $\mathcal{P}^{\log}(\Omega)$ the set of exponents satisfying the above conditions. If the exponent is in $\mathcal{P}^{\log}(\Omega)$ then $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$ for domains Ω with Lipschitz-continuous boundary. Let us now state some further results which will be needed later. The omitted proofs can be found for example in [KR91], [Die07], [DHHR10].

Theorem 1. *Let p be a bounded exponent in Ω . Then the mapping $I : L^{p'(\cdot)}(\Omega) \rightarrow (L^{p(\cdot)}(\Omega))^*$ with $\langle If, g \rangle := (f, g)$ is an isomorphism, and for all $f \in L^{p'(\cdot)}(\Omega)$ we have*

$$\frac{1}{2} \|f\|_{L^{p'(\cdot)}(\Omega)} \leq \|If\|_{(L^{p(\cdot)}(\Omega))^*} \leq 2 \|f\|_{L^{p'(\cdot)}(\Omega)}.$$

Theorem 2. *Let p, q and s be bounded exponents in Ω with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. For all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$ we have $fg \in L^{s(\cdot)}(\Omega)$ and*

$$\|fg\|_{L^{s(\cdot)}(\Omega)} \leq 2 \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{q(\cdot)}(\Omega)}.$$

Theorem 3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and p a bounded exponent in Ω . Then:*

- (1) *For every exponent q with $q \leq p$ a.e., the embedding*

$$L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$$

is continuous.

- (2) *If moreover $\partial\Omega$ is Lipschitz-continuous and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leq p^+ < \infty$, then the embedding*

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$$

is compact.

The following extension result can be found in [CUFMP06], [DF08], [DHHR10].

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Then there exists an exponent $\tilde{p} \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $\tilde{p}|_\Omega = p$ and $\tilde{p}^+ = p^+$, $\tilde{p}^- = p^-$ as well as an extension operator*

$$\mathcal{E} : W^{1,p(\cdot)}(\Omega) \rightarrow W^{1,\tilde{p}(\cdot)}(\mathbb{R}^n), \quad (\mathcal{E}f)|_\Omega = f.$$

When working with partial differential equations in unbounded domains, as we will have to later on, it is often not natural to assume that the solution and its derivatives belong to the same Lebesgue space. For this reason we now introduce the *homogeneous Sobolev spaces*. Let us present in the following the basic facts on those spaces. For details and proofs we refer to [Len08], [DHHR10].

For a bounded exponent p in Ω , and $k \in \mathbb{N}$ we define

$$\tilde{D}^{k,p(\cdot)}(\Omega) := \{u \in L^1_{\text{loc}}(\Omega) : \nabla^k u \in L^{p(\cdot)}(\Omega)\}.$$

The linear space $\tilde{D}^{k,p(\cdot)}(\Omega)$ is equipped with the seminorm

$$\|u\|_{\tilde{D}^{k,p(\cdot)}(\Omega)} := \|\nabla^k u\|_{L^{p(\cdot)}(\Omega)}.$$

Note, that $\|u\|_{\tilde{D}^{k,p(\cdot)}(\Omega)} = 0$ implies that u is a polynomial of degree $k-1$. Let us denote the polynomials of degree $m \in \mathbb{N}_0$ by \mathbf{P}_m . It is evident that the seminorm $\|\cdot\|_{\tilde{D}^{k,p(\cdot)}(\Omega)}$ becomes a norm on the equivalence classes $[u]$ defined for $u \in \tilde{D}^{k,p(\cdot)}(\Omega)$ by

$$[u]_{k-1} := \{w \in \tilde{D}^{k,p(\cdot)}(\Omega) : w = u + p_{k-1} \text{ for some } p_{k-1} \in \mathbf{P}_{k-1}\}.$$

Definition 5. Let p be a bounded exponent in Ω , and $k \in \mathbb{N}$. The homogeneous Sobolev space $D^{k,p(\cdot)}(\Omega)$ consists of all equivalence classes $[u]$ where $u \in \tilde{D}^{k,p(\cdot)}(\Omega)$. We identify u with its equivalence class $[u]$ and thus write u instead of $[u]$. The space $D^{k,p(\cdot)}(\Omega)$ is equipped with the norm

$$\|u\|_{D^{k,p(\cdot)}(\Omega)} := \|\nabla^k u\|_{L^{p(\cdot)}(\Omega)}.$$

Finally, we define the space $D_0^{k,p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{D^{k,p(\cdot)}(\Omega)}$.

Remark 6. The natural embedding $i: C_0^\infty(\Omega) \rightarrow D^{k,p(\cdot)}(\Omega): u \mapsto [u]$ implies that $C_0^\infty(\Omega)$ is isomorphic to a linear subspace of $D^{k,p(\cdot)}(\Omega)$. Consequently we can view $D_0^{k,p(\cdot)}(\Omega)$ as a subspace of $D^{k,p(\cdot)}(\Omega)$.

Theorem 7. The spaces $D^{k,p(\cdot)}(\Omega)$ and $D_0^{k,p(\cdot)}(\Omega)$ are separable Banach spaces which are reflexive if $1 < p^- \leq p^+ < \infty$.

For an integrable function u we define the mean value of u by $u_\Omega := \int_\Omega u \, dx$. The spaces $\tilde{D}^{k,p(\cdot)}(\Omega)$ and $W^{k,p(\cdot)}(\Omega)$ essentially do not differ for bounded domains. More precisely we have:

Theorem 8. Let Ω be a bounded domain with Lipschitz continuous boundary, and let $p \in \mathcal{P}^{\text{log}}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. Then we have the algebraic identity

$$\tilde{D}^{k,p(\cdot)}(\Omega) = W^{k,p(\cdot)}(\Omega).$$

Moreover for $u \in \tilde{D}^{1,p(\cdot)}(\Omega)$ we have the Poincaré inequality

$$(2.1) \quad \|u - u_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c \, \text{diam}(\Omega) \, \|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

with a constant c depending on n , the Lipschitz constant, and the log-Hölder constants of p .

Remark 9. As a consequence of the above theorem we get the algebraic identity

$$\tilde{D}^{k,p(\cdot)}(\Omega) = W_{\text{loc}}^{k,p(\cdot)}(\Omega)$$

for arbitray domains provided that $p \in \mathcal{P}^{\text{log}}$ satisfies $1 < p^- \leq p^+ < \infty$.

As for classical Sobolev spaces we have that $D^{k,p(\cdot)}(\mathbb{R}^n)$ and $D_0^{k,p(\cdot)}(\mathbb{R}^n)$ coincide.

Lemma 10. Let $p \in \mathcal{P}^{\text{log}}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$ and let $k \in \mathbb{N}_0$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $D^{k,p(\cdot)}(\mathbb{R}^n)$. Consequently we have $D^{k,p(\cdot)}(\mathbb{R}^n) = D_0^{k,p(\cdot)}(\mathbb{R}^n)$.

In applications it happens that for a function $u \in L^1_{\text{loc}}(\Omega)$ one can show that $\nabla u \in L^{p(\cdot)}(\Omega)$ and $\nabla^2 u \in L^{p(\cdot)}(\Omega)$. This information is neither covered by the space $D^{1,p(\cdot)}(\Omega)$ nor by the space $D^{2,p(\cdot)}(\Omega)$. Thus we introduce a new space containing the full information.

Definition 11. Let p be a bounded exponent in Ω . The space $D^{(1,2),p(\cdot)}(\Omega)$ consists of all equivalence classes $[u]_0$ with $u \in \tilde{D}^{1,p(\cdot)}(\Omega) \cap \tilde{D}^{2,p(\cdot)}(\Omega)$. We identify u with its equivalence class $[u]_0$ and thus write u instead of $[u]_0$. We equip the space $D^{(1,2),p(\cdot)}(\Omega)$ with the norm

$$\|u\|_{D^{(1,2),p(\cdot)}(\Omega)} := \|\nabla u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla^2 u\|_{L^{p(\cdot)}(\Omega)}.$$

Finally, we define the space $D_0^{(1,2),p(\cdot)}(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{D^{(1,2),p(\cdot)}(\Omega)}$.

Note that the space $D^{(1,2),p(\cdot)}(\Omega)$ is a subspace of $D^{1,p(\cdot)}(\Omega)$ but not of $D^{2,p(\cdot)}(\Omega)$, because it consists of equivalence classes modulo constants. As in Remark 6 one sees that $D_0^{(1,2),p(\cdot)}(\Omega)$ can be viewed as a subspace of $D^{(1,2),p(\cdot)}(\Omega)$.

Theorem 12. The spaces $D^{(1,2),p(\cdot)}(\Omega)$, and $D_0^{(1,2),p(\cdot)}(\Omega)$ are separable Banach spaces which are reflexive if $1 < p^- \leq p^+ < \infty$.

Lemma 13. Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$. Then $C_0^\infty(\mathbb{R}^n)$ is dense in $D^{(1,2),p(\cdot)}(\mathbb{R}^n)$. Consequently we have $D^{(1,2),p(\cdot)}(\mathbb{R}^n) = D_0^{(1,2),p(\cdot)}(\mathbb{R}^n)$.

We will also need the dual spaces of homogeneous Sobolev spaces.

Definition 14. Let p be a bounded exponent in Ω , and let $k \in \mathbb{N}_0$. The space $D^{-k,p(\cdot)}(\Omega)$ is defined as the dual of the space $D_0^{k,p'(\cdot)}(\Omega)$, i.e. $D^{-k,p(\cdot)}(\Omega) := (D_0^{k,p'(\cdot)}(\Omega))'$.

We let $L_0^{p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ be the subspace of functions having vanishing mean value. Analogously we define the subspace $C_{0,0}^\infty(\Omega) \subset C_0^\infty(\Omega)$.

Lemma 15. Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$, and $A \subset \Omega$ be a bounded domain with Lipschitz continuous boundary. Then $L_0^{p(\cdot)}(A) \hookrightarrow D^{-1,p(\cdot)}(\Omega)$ via $\langle f, u \rangle = \int_\Omega f u \, dx$ for $f \in L_0^{p(\cdot)}(A)$ and $u \in C_0^\infty(\Omega)$.

Proof. Using Theorem 8 we get

$$\begin{aligned} \langle f, u \rangle &= \int_\Omega f(x)(u(x) - u_A) \, dx \leq 2\|f\|_{L^{p(\cdot)}(\Omega)}\|u - u_A\|_{L^{p'(\cdot)}(A)} \\ (2.2) \quad &\leq c\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\nabla u\|_{L^{p'(\cdot)}(\Omega)}. \end{aligned}$$

□

If the domain Ω has a sufficiently large and nice boundary it is not necessary to require as in the previous lemma that the function f has a vanishing mean value. For simplicity we formulate the result only for the case of the half-space $\mathbb{R}_>^n := \{x \in \mathbb{R}^n | x_n > 0\}$. We define $\mathbb{R}_<^n$ accordingly and set $\Sigma := \partial\mathbb{R}_>^n$.

Lemma 16. Let $p \in \mathcal{P}^{\log}(\mathbb{R}_>^n)$ satisfy $1 < p^- \leq p^+ < \infty$ and $A \subset \mathbb{R}_>^n$ be a bounded domain. Then $L^{p(\cdot)}(A) \hookrightarrow D^{-1,p(\cdot)}(\mathbb{R}_>^n)$.

Proof. We choose a ball $B(x_0) \subset \mathbb{R}_>^n$ with $A \subset B(x_0)$ and $x_0 \in \Sigma$. Next we choose a ball $B' \subset \mathbb{R}_>^n \cap B(x_0)$ with $|B'| \approx |A|$ and note that $u_{B'} = 0$ for $u \in C_0^\infty(\mathbb{R}_>^n)$. Now we use a version of the Poincaré inequality (cf. [DHR10, Lemma 7.2.3 and Theorem 7.2.4 (b)]) to obtain for all $u \in C_0^\infty(\mathbb{R}_>^n)$

$$\begin{aligned} \langle f, u \rangle &= \int_{B(x_0)} f(x)u(x) \, dx \leq 2\|f\|_{L^{p(\cdot)}(B(x_0))}\|u - u_{B'}\|_{L^{p'(\cdot)}(B(x_0))} \\ (2.3) \quad &\leq c\|f\|_{L^{p(\cdot)}(A)}\|\nabla u\|_{L^{p'(\cdot)}(\mathbb{R}_>^n)}. \end{aligned}$$

□

Lemma 17. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. Then $C_{0,0}^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ and in $D^{-1,p(\cdot)}(\Omega)$.*

We will also have to deal with trace spaces of homogenous Sobolev spaces, at least in the case of the half-space $\mathbb{R}_>^n$. Traces are well defined for functions from $W_{\text{loc}}^{1,1}$ and thus the notion of a trace of a function from $\tilde{D}^{1,p(\cdot)}(\mathbb{R}_>^n)$ is well defined. Consequently, the *trace space* $\text{tr}(D^{1,p(\cdot)}(\mathbb{R}_>^n))$ consists of the (equivalence classes of) traces of all (equivalence classes) $f \in D^{1,p(\cdot)}(\mathbb{R}_>^n)$. The norm

$$\|f\|_{\text{tr}(D^{1,p(\cdot)}(\mathbb{R}_>^n))} := \inf \{ \|g\|_{D^{1,p(\cdot)}(\mathbb{R}_>^n)} : g \in D^{1,p(\cdot)}(\mathbb{R}_>^n) \text{ and } \text{tr } g = f \}$$

makes $\text{tr}(D^{1,p(\cdot)}(\mathbb{R}_>^n))$ a Banach space.

Another important issue is the extension of the theory of singular integrals to variable exponent spaces. We refer to [CUFMP06], [DHHR10] for proof of the following two theorems. Firstly, we will need the generalization of the classical Calderón-Zygmund theorem. We will restrict ourselves to symmetric kernels, i.e., to kernels, depending only on the difference of their arguments. A (symmetric) *kernel* K in Ω , $\Omega \subset \mathbb{R}^n$, is a locally integrable, real-valued function in $\Omega \setminus \{0\}$.

Theorem 18. *Let K be a kernel in \mathbb{R}^n of the form*

$$(2.4) \quad K(x) = \frac{P(x/|x|)}{|x|^n},$$

where $P \in L^r(\partial B_1(0))$ for some $r \in (1, \infty]$, and satisfying

$$(2.5) \quad \int_{\partial B_1(0)} P \, d\omega = 0.$$

Moreover let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ be bounded with $p^- > r'$. Then the operator T , defined by

$$(Tf)(x) := \lim_{\epsilon \searrow 0} \int_{(B_\epsilon(x))^c} K(x-y)f(y) \, dy$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Furthermore we will need the analogue of the famous Agmon-Douglis-Nirenberg result for spaces with variable exponents. To this end, let us fix some notation. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we set $x' := (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ and $\tilde{x} := (x_1, \dots, x_{n-1}, -x_n)$. Moreover we set $S := \partial B_1$, $S^{n-2} := \partial B_1^{n-1} \subset \mathbb{R}^{n-1}$, $S_> := S \cap \mathbb{R}_>^n$, and $S_\geq := S \cap \mathbb{R}_\geq^n$ where $\mathbb{R}_\geq^n := \{x \in \mathbb{R}^n | x_n \geq 0\}$.

Theorem 19. *Let K be a kernel on \mathbb{R}_\geq^n of the form*

$$K(x) = \frac{P(x/|x|)}{|x|^n},$$

where $P: S_\geq \rightarrow \mathbb{R}$ is continuous and satisfies

$$\int_{S^{n-2}} P(x', 0) \, d\omega' = 0.$$

Assume that K possesses continuous derivatives $\partial_i K$, $i = 1, \dots, n$, and $\partial_n^2 K$ in $\mathbb{R}_>^n$ which are bounded on the hemisphere $S_>$. Let $p \in \mathcal{P}^{\log}(\mathbb{R}_>^n)$ satisfy $1 < p^- \leq p^+ < \infty$. For $f \in C_0^\infty(\mathbb{R}_\geq^n)$ we define $Hf: \mathbb{R}_\geq^n \rightarrow \mathbb{R}$ by

$$(2.6) \quad (Hf)(x) := \int_{\Sigma} K(x' - y', x_n) f(y', 0) \, dy'.$$

Then H satisfies

$$(2.7) \quad \|\nabla H f\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \leq c \|\nabla f\|_{L^{p(\cdot)}(\mathbb{R}_>^n)},$$

for all $f \in C_0^\infty(\mathbb{R}_>^n)$ with a constant $c = c(p, n, P)$. In particular, H extends to a bounded linear operator $H: D^{1,p(\cdot)}(\mathbb{R}_>^n) \rightarrow D^{1,p(\cdot)}(\mathbb{R}_>^n)$.

Throughout the paper we will make use of Einsteins summation convention, i.e. whenever there is an index appearing twice in a monomial this implies that we are summing over all of its possible values.

3. STOKES SYSTEM

In this section we assume that Ω is a bounded domain in \mathbb{R}^n , $n \geq 2$, with $C^{1,1}$ -boundary. We want to show that the Stokes system

$$(3.1) \quad \begin{aligned} \Delta \mathbf{v} - \nabla \pi &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= g && \text{in } \Omega, \\ \mathbf{v} &= \mathbf{v}_0 && \text{on } \partial\Omega, \end{aligned}$$

possesses a unique strong and weak solution, respectively, provided that the data have appropriate regularity. More precisely we prove:

Theorem 20. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. For arbitrary data $\mathbf{f} \in (L^{p(\cdot)}(\Omega))^n$, $g \in W^{1,p(\cdot)}(\Omega)$ and $\mathbf{v}_0 \in \operatorname{tr}(W^{2,p(\cdot)}(\Omega))^n$ satisfying the compatibility condition $\int_\Omega g \, dx = \int_{\partial\Omega} \mathbf{v}_0 \cdot \boldsymbol{\nu} \, d\omega$ there exists a unique strong solution $(\mathbf{v}, \pi) \in (W^{2,p(\cdot)}(\Omega))^n \times W^{1,p(\cdot)}(\Omega)$ of the Stokes system (3.1) with $\int_\Omega \pi \, dx = 0$ and which satisfies the estimate*

$$\begin{aligned} \|\mathbf{v}\|_{W^{2,p(\cdot)}(\Omega)} + \|\pi\|_{W^{1,p(\cdot)}(\Omega)} \\ \leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\Omega)} + \|\mathbf{v}_0\|_{\operatorname{tr}(W^{2,p(\cdot)}(\Omega))} + \|g\|_{W^{1,p(\cdot)}(\Omega)} \right), \end{aligned}$$

where the constant c depends only on the domain Ω and the exponent p .

Theorem 21. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. For arbitrary data $\mathbf{f} \in (W^{-1,p(\cdot)}(\Omega))^n$, $g \in L^{p(\cdot)}(\Omega)$ and $\mathbf{v}_0 \in \operatorname{tr}((W^{1,p(\cdot)}(\Omega))^n)$ satisfying the above compatibility condition there exists a unique weak solution $(\mathbf{v}, \pi) \in (W^{1,p(\cdot)}(\Omega))^n \times L^{p(\cdot)}(\Omega)$ of the Stokes system (3.1) with $\int_\Omega \pi \, dx = 0$ and which satisfies the estimate*

$$\begin{aligned} \|\mathbf{v}\|_{W^{1,p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} \\ \leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)} + \|\mathbf{v}_0\|_{\operatorname{tr}(W^{1,p(\cdot)}(\Omega))} \right), \end{aligned}$$

where the constant c depends only on the domain Ω and the exponent p .

We call (\mathbf{v}, π) a *strong solution* of (3.1) provided that it satisfies the differential equations in (3.1) in the sense of weak derivatives. Furthermore we call (\mathbf{v}, π) a *weak solution* of (3.1) provided that

$$\begin{aligned} \int_\Omega \nabla \mathbf{v} : \nabla \phi \, dx - \int_\Omega \pi \cdot \operatorname{div} \phi \, dx &= -\langle \mathbf{f}, \phi \rangle \quad \forall \phi \in (W_0^{1,p'(\cdot)}(\Omega))^n, \\ \operatorname{div} \mathbf{v} &= g && \text{in } \Omega, \\ \mathbf{v} &= \mathbf{v}_0 && \text{on } \partial\Omega. \end{aligned}$$

In fact it is sufficient to consider homogeneous boundary conditions, i.e. $\mathbf{v}_0 = \mathbf{0}$. In the general case, when considering strong solutions, we take a realization $\tilde{\mathbf{v}}_0 \in (W^{2,p(\cdot)}(\Omega))^n$ of $\mathbf{v}_0 \in \operatorname{tr}((W^{2,p(\cdot)}(\Omega))^n)$ and construct a strong solution (\mathbf{v}, π) for the data $\mathbf{f} - \Delta \tilde{\mathbf{v}}_0$, $g - \operatorname{div} \tilde{\mathbf{v}}_0$ and vanishing boundary values. Note that $\int_\Omega (g - \operatorname{div} \tilde{\mathbf{v}}_0) \, dx = 0$. Then $(\mathbf{v} := \mathbf{u} + \tilde{\mathbf{v}}_0, \pi)$ is the unique strong solution, satisfying the assertions of the theorem. When dealing with weak solutions we may proceed in a very similar way.

Although the proof of Theorems 20 and 21 is based on the classical, i.e. constant exponent, theory of the Stokes system (cf. [Gal94]), we will nevertheless have to follow the strategy of the proof of the classical case all over again, i.e., we will use a localisation technique to reduce the problem in general bounded domains to the problem in the whole-space and in the half-space. The treatment of these situations is based on the Calderón-Zygmund theory of singular integral operators and on the Agmon-Douglis-Nirenberg theory of operators in the half-space.

It should be emphasized that deriving analogous results for the Poisson equation is much simpler than our task, although the proof essentially follows the same idea. The crucial difference is that the fundamental solution of the Poisson equation in half-space is given as the whole-space fundamental solution plus its own odd reflection. This simplifies the half-space case considerably. Indeed, the Agmon-Douglis-Nirenberg theory is not needed in the treatment of the Poisson equation. An analogous ansatz in the case of the Stokes system yields non-divergence free functions and thus won't work. This is the reason for the need of the Agmon-Douglis-Nirenberg theory and homogenous Sobolev spaces in this paper.

Since the structure of the fundamental solutions of the Stokes system is different for $n = 2$ and $n \geq 3$, we restrict ourselves to the latter case. The methods presented here can be easily adapted to treat also the case $n = 2$. Furthermore, using Theorem 4, a well behaved exponent given on a bounded domain will always be extended to the whole of \mathbb{R}^n without mentioning.

Solutions of the system

$$(3.2) \quad \begin{aligned} \Delta \mathbf{v} - \nabla \pi &= \mathbf{f} \quad \text{in } \mathbb{R}^n, \\ \operatorname{div} \mathbf{v} &= g \quad \text{in } \mathbb{R}^n, \end{aligned}$$

are obtained by a convolution of the *fundamental solutions* $(-\frac{1}{2|\partial B_1|}V^{rl})_{r,l=1,\dots,n}$ and $(-\frac{1}{|\partial B_1|}Q^l)_{l=1,\dots,n}$ of the Stokes system in the whole-space, given by

$$V^{rl}(x) := \left(\frac{1}{n-2} \frac{\delta_{rl}}{|x|^{n-2}} + \frac{x_r x_l}{|x|^n} \right) \quad \text{and} \quad Q^l(x) := \frac{x_l}{|x|^n},$$

with data \mathbf{f} and g . From the classical theory it is well known that the kernels $\partial_i \partial_j V^{rl}$ and $\partial_i Q^l$ satisfy the assumptions of Theorem 18. Consequently we get:

Lemma 22. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{f} \in (C_{0,0}^\infty(\mathbb{R}^n))^n$. Then the convolutions $\mathbf{v}(x) := \int_{\mathbb{R}^n} \mathbf{V}(x-y)\mathbf{f}(y)dy$ and $\pi(x) := \int_{\mathbb{R}^n} \mathbf{Q}(x-y) \cdot \mathbf{f}(y)dy$ are infinitely differentiable. Moreover, their first and second order derivatives have the representations $(i, j, r = 1, \dots, n)$*

$$\begin{aligned} \partial_i v_r(x) &= \int_{\mathbb{R}^n} \partial_{x_i} V^{rl}(x-y) f_l(y) dy, \\ \partial_i \partial_j v_r(x) &= \lim_{\epsilon \searrow 0} \int_{(B(x,\epsilon))^c} \partial_{x_i} \partial_{x_j} V^{rl}(x-y) f_l(y) dy \\ &\quad + \frac{2|B_1|}{n+2} \left(-(n+1)\delta_{ij} f_r(x) + \delta_{ir} f_j(x) + \delta_{rj} f_i(x) \right), \\ \partial_i \pi(x) &= \lim_{\epsilon \searrow 0} \int_{(B(x,\epsilon))^c} \partial_{x_i} Q^l(x-y) f_l(y) dy + |B_1| f_i(x), \end{aligned}$$

and satisfy the estimates

$$(3.3) \quad \begin{aligned} \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\pi\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)}, \\ \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\nabla \pi\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \|\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \end{aligned}$$

with a constant $c = c(p, n)$.

Proof. As in the classical theory (cf. [Gal94]) we deduce the representations of the derivatives using integration by parts. These representations together with Theorem 18 immediately yield the estimate (3.3)₂.

Using Theorem 1, the density of $C_0^\infty(B)$ in $L^{p(\cdot)}(B)$, the representation of $\nabla \mathbf{v}$ and the Theorem of Fubini we may estimate the norm of the first order derivatives of \mathbf{v} on every Ball $B \subset \mathbb{R}^n$ by

$$\begin{aligned} \|\partial_i v_r\|_{L^{p(\cdot)}(B)} &\leq 2 \sup_{\substack{\phi \in C_0^\infty(B), \\ \|\phi\|_{L^{p'(\cdot)}(B)} \leq 1}} \int_B \phi \partial_i v_r \, dx \\ &= 2 \sup_{\substack{\phi \in C_0^\infty(B), \\ \|\phi\|_{L^{p'(\cdot)}(B)} \leq 1}} \int_{\mathbb{R}^n} f_l(y) \underbrace{\int_{\mathbb{R}^n} \partial_{x_i} V^{rl}(x-y) \phi(x) \, dx}_{=:\Phi_l(y)} dy. \end{aligned}$$

From the properties of V and ϕ and from Theorem 18 we deduce the estimate

$$\|\nabla \Phi_l\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq c \|\phi\|_{L^{p'(\cdot)}(B)},$$

and thus

$$\|\partial_i v_r\|_{L^{p(\cdot)}(B)} \leq c \sup_{\substack{\Phi \in (D^{1,p(\cdot)}(\mathbb{R}^n))^n, \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} \mathbf{f} \cdot \Phi \, dy = c \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)}$$

with a constant c independent of B . The norm of π may be dealt with in the same way. Hence we get (3.3)₁. \square

Note that from the representations of the derivatives it follows that the convolutions $\mathbf{v}(x) := -\frac{1}{2|\partial B_1|} \int_{\mathbb{R}^n} \mathbf{V}(x-y) \mathbf{f}(y) dy$ and $\pi(x) := -\frac{1}{|\partial B_1|} \int_{\mathbb{R}^n} \mathbf{Q}(x-y) \cdot \mathbf{f}(y) dy$ solve the Stokes system in the whole-space, i.e. $\Delta \mathbf{v} - \nabla \pi = \mathbf{f}$, $\operatorname{div} \mathbf{v} = 0$ in \mathbb{R}^n .

Lemma 22 and the density of $C_{0,0}^\infty(\mathbb{R}^n)$ in $L^{p(\cdot)}(\mathbb{R}^n)$ and $D^{-1,p(\cdot)}(\mathbb{R}^n)$ (cf. Lemma 17) show that the convolution with the kernel \mathbf{V} extends to a bounded operator U from $(D^{-1,p(\cdot)}(\mathbb{R}^n))^n$ into $(D^{1,p(\cdot)}(\mathbb{R}^n))^n$ and from $(L^{p(\cdot)}(\mathbb{R}^n))^n$ into $(D^{2,p(\cdot)}(\mathbb{R}^n))^n$. Similarly we see that the convolution with the kernel \mathbf{Q} extends to a bounded operator P from $(D^{-1,p(\cdot)}(\mathbb{R}^n))^n$ into $L^{p(\cdot)}(\mathbb{R}^n)$ and from $(L^{p(\cdot)}(\mathbb{R}^n))^n$ into $D^{1,p(\cdot)}(\mathbb{R}^n)$. This proves the assertions (1) and (2) of the following theorem.

Theorem 23. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$ and U, P be the operators defined above.*

- (1) *If $\mathbf{f} \in (L^{p(\cdot)}(\mathbb{R}^n))^n$ then $U\mathbf{f} \in (D^{2,p(\cdot)}(\mathbb{R}^n))^n$ and $P\mathbf{f} \in D^{1,p(\cdot)}(\mathbb{R}^n)$ satisfy the estimate*

$$\|U\mathbf{f}\|_{D^{2,p(\cdot)}(\mathbb{R}^n)} + \|P\mathbf{f}\|_{D^{1,p(\cdot)}(\mathbb{R}^n)} \leq c \|\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with a constant $c = c(p, n)$.

- (2) *If $\mathbf{f} \in (D^{-1,p(\cdot)}(\mathbb{R}^n))^n$ then $U\mathbf{f} \in (D^{1,p(\cdot)}(\mathbb{R}^n))^n$ and $P\mathbf{f} \in L^{p(\cdot)}(\mathbb{R}^n)$ satisfy the estimate*

$$\|U\mathbf{f}\|_{D^{1,p(\cdot)}(\mathbb{R}^n)} + \|P\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)}$$

with a constant $c = c(p, n)$.

- (3) *If $\mathbf{f} \in (L^{p(\cdot)}(\mathbb{R}^n))^n$ has bounded support and vanishing mean value, hence $\mathbf{f} \in (D^{-1,p(\cdot)}(\mathbb{R}^n))^n$, then $U\mathbf{f} \in (D^{(1,2),p(\cdot)}(\mathbb{R}^n))^n \subset (D^{1,p(\cdot)}(\mathbb{R}^n))^n$ and $P\mathbf{f} \in W^{1,p(\cdot)}(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$ satisfy both of the above estimates simultaneously.*

Proof. It remains to show (3). We consider the operators U and P from assertion (2). Due to the estimate (2.2) any \mathbf{f} as in (3) approximated by some sequence $(\mathbf{f}_k) \subset (C_{0,0}^\infty(\mathbb{R}^n))^n$ in $(L^{p(\cdot)}(\mathbb{R}^n))^n$ is approximated by the same sequence

in $(D^{-1,p(\cdot)}(\mathbb{R}^n))^n$. From (1) and (2) thus follows that $(P\mathbf{f}_k)$ is a Cauchy sequence in $W^{1,p(\cdot)}(\mathbb{R}^n)$ while $(U\mathbf{f}_k)$ is a Cauchy sequence in $(D^{(1,2),p(\cdot)})^n$. \square

Corollary 24. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{f} \in (L^{p(\cdot)}(\mathbb{R}^n))^n$ and $g \in W^{1,p(\cdot)}(\mathbb{R}^n)$ have bounded support. Moreover, let \mathbf{f} have vanishing mean value and let $(\mathbf{v}, \pi) \in (W^{2,p^-}(\mathbb{R}^n))^n \times W^{1,p^-}(\mathbb{R}^n)$ be a solution of the Stokes system (3.2). Then the first and second weak derivatives of \mathbf{v} as well as π and its first weak derivatives belong to the space $L^{p(\cdot)}(\mathbb{R}^n)$. They satisfy the estimates*

$$(3.4) \quad \begin{aligned} \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\pi\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \left(\|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right), \\ \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\nabla \pi\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}^n)} + \|\nabla g\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right), \end{aligned}$$

with a constant $c = c(p, n)$.

Proof. From Section 4 we know that the function

$$\mathbf{h} := -L(\nabla g) \in ((D^{(1,2),p(\cdot)})^n$$

satisfies the estimates

$$(3.5) \quad \begin{aligned} \|\nabla \mathbf{h}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \\ \|\nabla^2 \mathbf{h}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq c \|\nabla g\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \end{aligned}$$

and the identities $\Delta \mathbf{h} = \nabla g$ und $\operatorname{div} \mathbf{h} = g$ in \mathbb{R}^n . Note that ∇g has vanishing mean value and $\|\nabla g\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} \leq \|g\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Set $\mathbf{F} := \mathbf{f} - \Delta \mathbf{h}$. The remark following Lemma 22 and a density argument imply that

$$\tilde{\mathbf{u}} := -\frac{1}{2|\partial B_1|} U \mathbf{F}$$

and

$$\tilde{\pi} := -\frac{1}{|\partial B_1|} P \mathbf{F}$$

solve the Stokes system in the whole-space, i.e. $\Delta \tilde{\mathbf{u}} - \nabla \tilde{\pi} = \mathbf{F}$, $\operatorname{div} \tilde{\mathbf{u}} = 0$ in \mathbb{R}^n . Setting $\tilde{\mathbf{v}} := \tilde{\mathbf{u}} + \mathbf{h}$ we conclude that

$$\begin{aligned} \Delta \tilde{\mathbf{v}} - \nabla \tilde{\pi} &= \Delta \tilde{\mathbf{u}} + \Delta \mathbf{h} - \nabla \tilde{\pi} = \nabla \tilde{\pi} + \mathbf{F} + \Delta \mathbf{h} - \nabla \tilde{\pi} = \mathbf{f}, \\ \operatorname{div} \tilde{\mathbf{v}} &= g, \end{aligned}$$

in \mathbb{R}^n . From theorem 23 and (3.5) we easily deduce the estimates (3.4) for \mathbf{v} and π replaced by $\tilde{\mathbf{v}}$ and $\tilde{\pi}$.

Since \mathbf{f} and g have bounded support analogous estimates hold with $p(\cdot)$ replaced by p^- . Using these estimates, the fact that solutions $(\mathbf{v}, \pi) \in D^{1,q}(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$, $1 < q < \infty$, of the Stokes system (3.2) are unique up to a constant (cf. [Gal94, Theorem IV.2.2]), and the integrability of π and $\tilde{\pi}$ we obtain (3.4). \square

Now we are ready to prove interior estimates for solutions of the Stokes system.

Theorem 25. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{f} \in (L^{p(\cdot)}(\Omega))^n$ and $g \in W^{1,p(\cdot)}(\Omega)$. Let $\Omega_0, \Omega_1 \subset \Omega$ be open sets with $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$. Moreover, let $(\mathbf{v}, \pi) \in (W^{2,p(\cdot)}(\Omega))^n \times W^{1,p(\cdot)}(\Omega)$ be a solution of the Stokes system (3.1)_{1,2}. Then there exists a constant $c = c(p, \Omega_0, \Omega_1)$ such that (\mathbf{v}, π) satisfy the estimates*

$$\begin{aligned} &\|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\Omega_0)} + \|\pi\|_{L^{p(\cdot)}(\Omega_0)} \\ &\leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega_1)} + \|g\|_{L^{p(\cdot)}(\Omega_1)} + \|\mathbf{v}\|_{L^{p(\cdot)}(\Omega_1 \setminus \Omega_0)} + \|\pi\|_{W^{-1,p(\cdot)}(\Omega_1 \setminus \Omega_0)} \right), \\ &\|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\Omega_0)} + \|\nabla \pi\|_{L^{p(\cdot)}(\Omega_0)} \\ &\leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\Omega_1)} + \|g\|_{W^{1,p(\cdot)}(\Omega_1)} + \|\mathbf{v}\|_{W^{1,p(\cdot)}(\Omega_1 \setminus \Omega_0)} + \|\pi\|_{L^{p(\cdot)}(\Omega_1 \setminus \Omega_0)} \right). \end{aligned}$$

Proof. Let $\tau \in C^\infty(\mathbb{R}^n)$ with $\tau = 1$ in Ω_0 und $\text{supp}(\tau) \subset\subset \Omega_1$. For $\bar{\mathbf{v}} := \mathbf{v}\tau$ and $\bar{\pi} := \pi\tau$ we have

$$\begin{aligned}\Delta \bar{\mathbf{v}} - \nabla \bar{\pi} &= 2\nabla \mathbf{v} \nabla \tau + \Delta \tau \mathbf{v} - \pi \nabla \tau + \mathbf{f}\tau =: \mathbf{T}, \\ \text{div } \bar{\mathbf{v}} &= \mathbf{v} \cdot \nabla \tau + g\tau =: G,\end{aligned}$$

in \mathbb{R}^n . Integrating by parts we obtain

$$\int_{\mathbb{R}^n} \mathbf{T} \, dx = \int_{\mathbb{R}^n} (-\Delta \mathbf{v} + \nabla \pi + \mathbf{f}) \phi \, dx = 0.$$

Hence we see that $\bar{\mathbf{v}} \in (W^{2,p(\cdot)}(\mathbb{R}^n))^n$, $\bar{\pi} \in W^{1,p(\cdot)}(\mathbb{R}^n)$, $G \in W^{1,p(\cdot)}(\mathbb{R}^n)$ and $\mathbf{T} \in (L^{p(\cdot)}(\mathbb{R}^n))^n$ satisfy the assumptions of corollary 24 which yields

$$\begin{aligned}\|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\Omega_0)} + \|\pi\|_{L^{p(\cdot)}(\Omega_0)} &\leq c \left(\|\mathbf{T}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} + \|G\|_{L^{p(\cdot)}(\Omega_1)} \right), \\ \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\Omega_0)} + \|\nabla \pi\|_{L^{p(\cdot)}(\Omega_0)} &\leq c \left(\|\mathbf{T}\|_{L^{p(\cdot)}(\Omega_1)} + \|\nabla G\|_{L^{p(\cdot)}(\Omega_1)} \right).\end{aligned}$$

The last estimate immediately yields the second estimate of the theorem. To prove the first one we use

$$\begin{aligned}\|\mathbf{T}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} &= \sup_{\substack{\Psi \in D_0^{1,p'(\cdot)}(\mathbb{R}^n), \\ \|\nabla \Psi\|_{p'(\cdot),\mathbb{R}^n} \leq 1}} \int_{\mathbb{R}^n} (-\mathbf{v} \cdot \Psi \Delta \tau - 2\mathbf{v} \cdot \nabla \Psi \nabla \tau - \pi \nabla \tau \cdot \Psi + \mathbf{f} \cdot \Psi \tau) \, dx \\ &\leq c \sup_{\substack{\Psi \in D_0^{1,p'(\cdot)}(\mathbb{R}^n), \\ \|\nabla \Psi\|_{p'(\cdot),\mathbb{R}^n} \leq 1}} \left(\|\mathbf{v}\|_{L^{p(\cdot)}(\Omega_1 \setminus \Omega_0)} \|\Psi \Delta \tau\|_{L^{p'(\cdot)}(\Omega_1 \setminus \Omega_0)} \right. \\ &\quad + \|\mathbf{v}\|_{L^{p(\cdot)}(\Omega_1 \setminus \Omega_0)} \|\nabla \Psi \nabla \tau\|_{L^{p'(\cdot)}(\Omega_1 \setminus \Omega_0)} \\ &\quad + \|\pi\|_{W^{-1,p(\cdot)}(\Omega_1 \setminus \Omega_0)} \|\Psi \cdot \nabla \tau\|_{W^{1,p'(\cdot)}(\Omega_1 \setminus \Omega_0)} \\ &\quad \left. + \|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega_1)} \|\Psi \tau\|_{W^{1,p'(\cdot)}(\Omega_1)} \right).\end{aligned}$$

We integrated by parts to derive the equality. Afterwards we restricted the domains of integration appropriately and used Hölder's inequality. Since \mathbf{T} has vanishing mean value we may assume that Ψ has vanishing mean value in Ω_1 . Hence, using theorem 8 we may estimate the $p'(\cdot)$ -norm of the cut-off Ψ in Ω_1 by $\|\nabla \Psi\|_{p'(\cdot),\mathbb{R}^n}$. This finishes the proof. \square

Now we turn our attention to the *Stokes system in the half-space*

$$\begin{aligned}(3.6) \quad \Delta \mathbf{v} - \nabla \pi &= \mathbf{f} \quad \text{in } \mathbb{R}_>^n, \\ \text{div } \mathbf{v} &= g \quad \text{in } \mathbb{R}_>^n, \\ \mathbf{v} &= \mathbf{0} \quad \text{on } \Sigma.\end{aligned}$$

In order to derive estimates for this problem we reflect the data in an even manner and, by a convolution with the fundamental solutions of the Stokes system, produce a whole-space solution $\tilde{\mathbf{v}}$ of (3.2). This solution does not satisfy the homogeneous boundary condition $\tilde{\mathbf{v}} = \mathbf{0}$ on Σ . To achieve this we add to $\tilde{\mathbf{v}}$ a solution of the problem in the half-space

$$\begin{aligned}(3.7) \quad \Delta \mathbf{w} - \nabla \nu &= \mathbf{0} \quad \text{in } \mathbb{R}_>^n, \\ \text{div } \mathbf{w} &= 0 \quad \text{in } \mathbb{R}_>^n, \\ \mathbf{w} &= \mathbf{h} \quad \text{on } \Sigma,\end{aligned}$$

with the special choice $\mathbf{h} = -\tilde{\mathbf{v}}|_\Sigma$. In order to obtain appropriate estimates of solutions of (3.7) we need Theorem 19. The solutions of this problem are obtained

as usual by a convolution of the normal derivatives of the fundamental solutions in the half-space, namely $\mathbf{Z} = (Z^{rl})_{r,l=1,\dots,n}$ and $(\partial_l z)_{l=1,\dots,n}$ with

$$Z^{rl}(x) := \frac{2}{|B_1|} \frac{x_n x_r x_l}{|x|^{n+2}},$$

and

$$z(x) := -\frac{4}{|\partial B_1|} \frac{x_n}{|x|^n},$$

with the boundary data \mathbf{h} from (3.7). From the classical theory it is well known that the kernels Z^{rl} and z satisfy the assumptions of Theorem 19. Thus we obtain:

Theorem 26. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{h} \in (D^{(1,2),p(\cdot)}(\mathbb{R}^n))^n$. Then there exists a solution*

$$(\mathbf{w}, \nu) \in (D^{(1,2),p(\cdot)}(\mathbb{R}_>^n))^n \times W^{1,p(\cdot)}(\mathbb{R}_>^n)$$

of the Stokes system in the half-space (3.7) with boundary data $\mathbf{h}|_\Sigma$ which satisfies the estimates

$$(3.8) \quad \begin{aligned} \|\nabla \mathbf{w}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nu\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \|\nabla \mathbf{h}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)}, \\ \|\nabla^2 \mathbf{w}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla \nu\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \|\nabla^2 \mathbf{h}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)}, \end{aligned}$$

with a constant $c = c(p, n)$.

Proof. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $D^{(1,2),p(\cdot)}(\mathbb{R}^n)$ it suffices to show the existence of a linear solution operator defined on $C_0^\infty(\mathbb{R}^n)$ and satisfying the estimates (3.8). Thus let $\mathbf{h} \in (C_0^\infty(\mathbb{R}^n))^n$. Then the functions $\mathbf{w} : \mathbb{R}_>^n \rightarrow \mathbb{R}^n$ and $s : \mathbb{R}_>^n \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \mathbf{w}(x) &:= \int_{\Sigma} \mathbf{Z}(x' - y', x_n) \mathbf{h}(y', 0) \, dy', \\ \nu(x) &:= \underbrace{\sum_{i=1}^n \partial_i \int_{\Sigma} z(x' - y', x_n) h_i(y', 0) \, dy'}_{=: \nu_i(x)}, \end{aligned}$$

are smooth solutions of (3.7). Indeed, it is easy to see that \mathbf{w} and ν are infinitely differentiable and that the derivatives may be written as

$$(3.9) \quad \begin{aligned} \partial^\alpha \mathbf{w}(x) &= \int_{\Sigma} \partial_x^\alpha \mathbf{Z}(x - (y', 0)) \mathbf{f}(y') \, dy', \\ \partial^\alpha \nu(x) &= \int_{\Sigma} \partial_x^\alpha \partial_{x_i} z(x - (y', 0)) f_i(y') \, dy'. \end{aligned}$$

Using these representations and the fact that $\Delta Z^{rl} - \partial_{rl}^2 z = 0$ and $\partial_r Z^{rl} = 0$ in $\mathbb{R}^n \setminus \{0\}$ we conclude that \mathbf{w} and ν solve the Stokes system (3.7)_{1,2}. In order to show that the boundary values are met continuously we assume that $x \in \mathbb{R}_>^n$ and $\epsilon > 0$. It is not hard to see that

$$\int_{B_\epsilon^{n-1}(x')} Z^{rl}(x - (y', 0)) \, dy' = \delta_{rl} + o(1)$$

for $x_n \searrow 0$. Thereby we get

$$\begin{aligned} w_r(x) - f_r(x') &= \int_{B_\epsilon^{n-1}(x')} Z^{rl}(x - (y', 0)) f_l(y') dy' - f_r(x') \\ &\quad + \int_{\Sigma \setminus B_\epsilon^{n-1}(x')} Z^{rl}(x - (y', 0)) f_l(y') dy' \\ &= \int_{B_\epsilon^{n-1}(x')} Z^{rl}(x - (y', 0)) (f_l(y') - f_l(x')) dy' + o(1) \end{aligned}$$

for $x_n \searrow 0$, since for the second summand we get by using Hölder's inequality

$$\begin{aligned} \left| \int_{\Sigma \setminus B_\epsilon^{n-1}(x')} Z^{rl}(x - (y', 0)) f_l(y') dy' \right| &\leq x_n c \int_{\Sigma \setminus B_\epsilon^{n-1}(x')} \frac{|f_l(y')|}{|x' - y'|^n} dy' \\ &= x_n c(\epsilon) \in o(1) \text{ for } x_n \searrow 0. \end{aligned}$$

Thus we may conclude that

$$\begin{aligned} |w_r(x) - f_r(x')| &\leq \underbrace{\sup_{y' \in B_\epsilon^{n-1}(x')} |\mathbf{f}(y') - \mathbf{f}(x')|}_{\leq \delta, \text{ if } \epsilon \text{ is suff. small}} \sum_l \int_{B_\epsilon^{n-1}(x')} |Z^{rl}(x - (y', 0))| dy' + o(1) \\ &\leq c\delta + o(1) \end{aligned}$$

for $x_n \searrow 0$ and arbitrary $\delta > 0$. It is not hard to see that the constant c is independent of x_n . Hence we get

$$\limsup_{x_n \searrow 0} |w_r(x) - f_r(x')| \leq c\delta.$$

This finishes the proof that \mathbf{w} and ν are smooth solutions.

The estimate (3.8)₁ now follows from Theorem 19 applied to \mathbf{w} and ν_i , $i = 1, \dots, n$. In order to prove (3.8)₂ we notice that for $1 \leq k < n$ we have

$$\begin{aligned} \partial_k \mathbf{w}(x) &= \int_{\Sigma} \mathbf{Z}(x' - y', x_n) \partial_k \mathbf{h}(y', 0) dy', \\ \partial_k \nu(x) &= \partial_i \underbrace{\int_{\Sigma} z(x' - y', x_n) \partial_k h_i(y', 0) dy'}_{=: \nu_{ik}(x)}. \end{aligned}$$

Again Theorem 19 applied to $\partial_k \mathbf{w}$ and ν_{ik} gives

$$\|\partial_k \nabla \mathbf{w}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\partial_k \nu\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \leq c \|\nabla^2 \mathbf{h}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)}.$$

Using the equations (3.7)_{1,2} we compute $\partial_n^2 w_n = -\sum_{i < n} \partial_{ni}^2 w_i$, $\partial_n^2 w_i = \partial_i \nu - \sum_{j < n} \partial_j^2 w_i$, $1 \leq i < n$, and $\partial_n \nu = \Delta w_n$. These identities together with the last estimate give the estimate for $\partial_n^2 \mathbf{w}$ and $\partial_n \nu$. This finishes the proof of the theorem. \square

Using Corollary 24 and the previous theorem we get half-space estimates for the Stokes system (3.6).

Corollary 27. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}_>^n)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{f} \in (L^{p(\cdot)}(\mathbb{R}_>^n))^n$ and $g \in W^{1,p(\cdot)}(\mathbb{R}_>^n)$ have bounded support. Let $(\mathbf{v}, \pi) \in (W^{2,p^-}(\mathbb{R}_>^n))^n \times W^{1,p^-}(\mathbb{R}_>^n)$ be a solution of the Stokes system in the half-space (3.6) corresponding to the data*

\mathbf{f} and g . Then the first and the second weak derivatives of \mathbf{v} as well as π and its first derivatives belong to the space $L^{p(\cdot)}(\mathbb{R}_>^n)$. They satisfy the estimates

$$(3.10) \quad \begin{aligned} \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\pi\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \left(\|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}_>^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \right), \\ \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla \pi\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla g\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \right), \end{aligned}$$

with a constant $c = c(p, n)$.

Proof. Note that $\mathbf{f} \in D^{-1,p(\cdot)}(\mathbb{R}_>^n)$ by Lemma 16. We extend p and g by an even reflection, and \mathbf{f} by an odd reflection to \mathbb{R}^n . Thus $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $\mathbf{f} \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in W^{1,p(\cdot)}(\mathbb{R}^n)$ with corresponding estimates of the whole-space norms by the half-space norms. Moreover, \mathbf{f} has vanishing mean value, and \mathbf{f} and g still have bounded support. We now construct the whole-space solution $(\tilde{\mathbf{v}}, \tilde{\pi})$ corresponding to this data in the same way as in the proof of Corollary 24. Thus we get

$$(3.11) \quad \begin{aligned} \|\nabla \tilde{\mathbf{v}}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\tilde{\pi}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \left(\|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}_>^n)} + \|g\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \right), \\ \|\nabla^2 \tilde{\mathbf{v}}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla \tilde{\pi}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla g\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} \right), \end{aligned}$$

where we used that $\|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} \leq \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}_>^n)}$. This estimate may be shown in the following way. Every function ϕ in \mathbb{R}^n can be split into an even and an odd part:

$$\phi(x) = \frac{\phi(x) + \phi(\tilde{x})}{2} + \frac{\phi(x) - \phi(\tilde{x})}{2} =: \phi_e(x) + \phi_o(x).$$

Since \mathbf{f} is odd, we have $\int_{\mathbb{R}^n} \mathbf{f}_l \phi = \int_{\mathbb{R}^n} \mathbf{f}_l \phi_o$. Hence we get

$$\begin{aligned} \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)} &= \sup_{\substack{\Phi \in C_0^\infty(\mathbb{R}^n), \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} \mathbf{f} \cdot \Phi \, dx = \sup_{\substack{\Phi \in C_0^\infty(\mathbb{R}^n), \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} \mathbf{f} \cdot \Phi_o \, dx \\ &\leq \sup_{\substack{\Phi \in C_0^\infty(\mathbb{R}^n), \text{ odd}, \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq 1}} \int_{\mathbb{R}^n} \mathbf{f} \cdot \Phi \, dx = 2 \sup_{\substack{\Phi \in C_0^\infty(\mathbb{R}^n), \text{ odd}, \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}_>^n)} \leq 1}} \int_{\mathbb{R}_>^n} \mathbf{f} \cdot \Phi \, dx \\ &\leq \sup_{\substack{\Phi \in C_0^\infty(\mathbb{R}_>^n), \\ \|\nabla \Phi\|_{L^{p'(\cdot)}(\mathbb{R}_>^n)} \leq 1}} \int_{\mathbb{R}_>^n} \mathbf{f} \cdot \Phi \, dx = \|\mathbf{f}\|_{D^{-1,p(\cdot)}(\mathbb{R}_>^n)}. \end{aligned}$$

The second inequality holds because every odd function $\phi \in C_0^\infty(\mathbb{R}^n)$ is in $D_0^{1,p'(\cdot)}(\mathbb{R}_>^n)$ and $\|\nabla \phi\|_{p'(\cdot), \mathbb{R}^n} \leq 2\|\nabla \phi\|_{p'(\cdot), \mathbb{R}_>^n}$.

Theorem 26 yields the existence of a solution (\mathbf{w}, ν) of the Stokes system in the half-space (3.7) with boundary data $-\tilde{\mathbf{v}}|_\Sigma$ satisfying the estimates

$$\begin{aligned} \|\nabla \mathbf{w}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nu\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \|\nabla \tilde{\mathbf{v}}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)}, \\ \|\nabla^2 \mathbf{w}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} + \|\nabla \nu\|_{L^{p(\cdot)}(\mathbb{R}_>^n)} &\leq c \|\nabla^2 \tilde{\mathbf{v}}\|_{L^{p(\cdot)}(\mathbb{R}_>^n)}. \end{aligned}$$

These estimates together with (3.11) imply that $\bar{\mathbf{v}} := \tilde{\mathbf{v}} + \mathbf{w}$ and $\bar{\pi} := \tilde{\pi} + \nu$ satisfy the estimates (3.10) and solve the problem

$$\begin{aligned} \Delta \bar{\mathbf{v}} &= \Delta \tilde{\mathbf{v}} + \Delta \mathbf{w} = \mathbf{f} + \nabla \bar{\pi} && \text{in } \mathbb{R}_>^n, \\ \operatorname{div} \bar{\mathbf{v}} &= g && \text{in } \mathbb{R}_>^n, \\ \bar{\mathbf{v}} &= \tilde{\mathbf{v}} - \tilde{\mathbf{v}} = \mathbf{0} && \text{on } \Sigma. \end{aligned}$$

If we replace $p(\cdot)$ by p^- in the above arguments we get that $\bar{\mathbf{v}}$ and $\bar{\pi}$ also satisfy the corresponding estimates (3.10) with $p(\cdot)$ replaced by p^- . Using the classical uniqueness result [Gal94, Theorem IV.3.3] we deduce that also \mathbf{v} and π satisfy the estimates (3.10). \square

Now we are ready to prove estimates near the boundary for solutions of the Stokes system (3.1) with homogeneous boundary data provided that the boundary is of class $C^{1,1}$.

Definition 28. We say that a domain $\Omega \subset \mathbb{R}^n$ has a $C^{1,1}$ -boundary if for every boundary point $\bar{x} \in \partial\Omega$ there is a rotation and translation G of \mathbb{R}^n and a function $a \in C^{1,1}$ such that $G(0) = \bar{x}$, $a(0) = \nabla a(0) = 0$ and

$$\begin{aligned}\Lambda &:= G(\{(x', x_n) \in \mathbb{R}^n \mid |x'| < \alpha, a(x') = x_n\}) \subset \partial\Omega, \\ V &:= G(\{(x', x_n) \in \mathbb{R}^n \mid |x'| < \alpha, a(x') < x_n < a(x') + \beta\}) \subset \Omega, \\ V_- &:= G(\{(x', x_n) \in \mathbb{R}^n \mid |x'| < \alpha, a(x') - \beta < x_n < a(x')\}) \subset \mathbb{R}^n \setminus \bar{\Omega},\end{aligned}$$

for some $\alpha, \beta > 0$.

Note that in fact the assumption $a(0) = \nabla a(0) = 0$ is not a restriction since this simply means that we describe the boundary as the graph of function defined on the tangential space.

Theorem 29. Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $\mathbf{f} \in (L^{p(\cdot)}(\Omega))^n$ and $g \in W^{1,p(\cdot)}(\Omega)$. Let $(\mathbf{v}, \pi) \in (W^{2,p(\cdot)}(\Omega))^n \times W^{1,p(\cdot)}(\Omega)$ be a solution of the Stokes system (3.1) with $\mathbf{v}_0 = \mathbf{0}$. Moreover we fix a boundary point $\bar{x} \in \partial\Omega$ and consider the corresponding set V from the previous definition. Moreover define V' analogously to V with α, β replaced by α', β' where $0 < \alpha' < \alpha, 0 < \beta' < \beta$. If α is chosen sufficiently small there exists a constant $c = c(p, V, V', \Omega)$ such that (\mathbf{v}, π) satisfy the estimates

$$\begin{aligned}(3.12) \quad & \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(V')} + \|\pi\|_{L^{p(\cdot)}(V')} \\ & \leq c(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(V)} + \|g\|_{L^{p(\cdot)}(V)} + \|\mathbf{v}\|_{L^{p(\cdot)}(V)} + \|\pi\|_{W^{-1,p(\cdot)}(V)}), \\ & \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(V')} + \|\nabla \pi\|_{L^{p(\cdot)}(V')} \\ & \leq c(\|\mathbf{f}\|_{L^{p(\cdot)}(V)} + \|g\|_{W^{1,p(\cdot)}(V)} + \|\mathbf{v}\|_{W^{1,p(\cdot)}(V)} + \|\pi\|_{L^{p(\cdot)}(V)}).\end{aligned}$$

Proof. For simplicity we assume that the possible rotation and translation is not present, i.e. $G = Id$. We define V'' analogously to V' with $0 < \alpha' < \alpha'' < \alpha$ and $0 < \beta' < \beta'' < \beta$. Let $\tau \in C^\infty(\bar{\Omega})$ satisfy $\tau = 1$ in V' and $\tau = 0$ outside of V'' . Let us straighten the boundary with the help of the coordinate transformation $\mathbf{F} : V \rightarrow \mathbf{F}(V) =: \hat{V} \subset \mathbb{R}^n$, where $(y', y_n) := \mathbf{F}(x', x_n) := (x', x_n - a(x'))$ (cf. Figure 3.1).

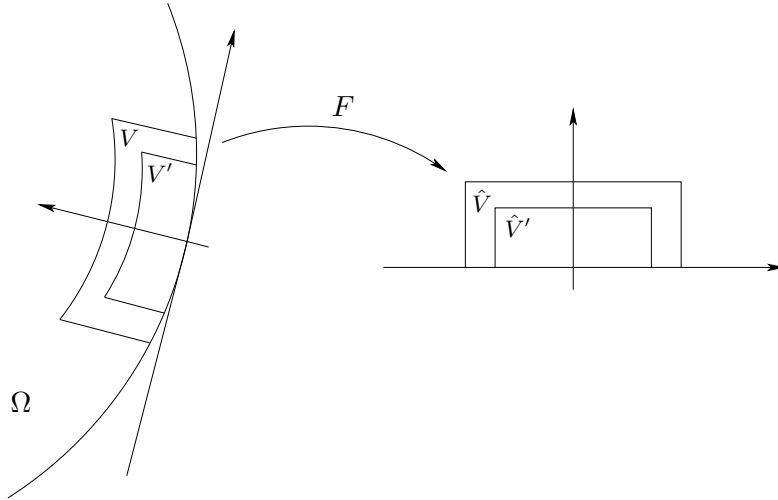


FIGURE 3.1.

We set $\widehat{V}' := \mathbf{F}(V')$, $\widehat{\tau} := \tau \circ \mathbf{F}^{-1}$, and analogously $\widehat{\mathbf{v}}, \widehat{\pi}, \widehat{\mathbf{f}}, \widehat{g}$ and \widehat{p} . Note that $\widehat{p} \in \mathcal{P}^{\log}(\widehat{V})$. Furthermore we define $\widehat{\mathbf{v}} := \widehat{\mathbf{v}}\widehat{\tau} \in (W^{2,\widehat{p}(\cdot)}(\mathbb{R}_>^n))^n$ and $\widehat{\pi} := \widehat{\pi}\widehat{\tau} \in W^{1,\widehat{p}(\cdot)}(\mathbb{R}_>^n)$. The integrabilities may be seen by using the identity

$$(3.13) \quad \|\widehat{f}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} = \|f\|_{L^{p(\cdot)}(V)}.$$

A tedious but simple computation shows that the couple $(\widehat{\mathbf{v}}, \widehat{\pi})$ solves the Stokes system in the half-space (3.6) with data (\mathbf{T}, G) defined by

$$\begin{aligned} T_j &:= \widehat{\tau} \widehat{f}_j + A_i \partial_{in}^2 \widehat{v}_j + B_i \partial_i \widehat{v}_j + C \widehat{v}_j + D_j \partial_n \widehat{\pi} + E_j \widehat{\pi}, \\ G &:= \widehat{\tau} \widehat{g} + R_i \widehat{v}_i + S_i \partial_n \widehat{v}_i, \end{aligned}$$

where B_i, C, E_j, R_i are bounded functions depending on τ, a and their derivatives of up to second order. Note that $a \in W^{2,\infty}$. Moreover A_i, D_j and S_i are scalings of first order derivatives of a , and thus can be made arbitrarily small by reducing α . Hence we will be able to absorb the corresponding terms in the left hand sides of the estimates we are about to derive. Since $(\mathbf{T}, G) \in (L^{\widehat{p}(\cdot)}(\mathbb{R}_>^n))^n \times W^{1,\widehat{p}(\cdot)}(\mathbb{R}_>^n)$ have bounded support we can use Corollary 27 to obtain

$$\begin{aligned} \|\nabla^2 \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\nabla \widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \\ \leq c \left(\|\widehat{\mathbf{f}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{g}\|_{W^{1,\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\mathbf{v}}\|_{W^{1,\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right) \\ + \frac{1}{2} \left(\|\nabla^2 \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\nabla \widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right). \end{aligned}$$

Absorbing into the left hand side we get

$$\begin{aligned} \|\nabla^2 \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V}')} + \|\nabla \widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V}')} &\leq \|\nabla^2 \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\nabla \widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \\ &\leq c \left(\|\widehat{\mathbf{f}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{g}\|_{W^{1,\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\mathbf{v}}\|_{W^{1,\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right). \end{aligned}$$

Transforming back via \mathbf{F}^{-1} we derive the estimate (3.12)₂. Corollary 27 also gives

$$\|\nabla \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \leq c \left(\|\mathbf{T}\|_{D^{-1,\widehat{p}(\cdot)}(\mathbb{R}_>^n)} + \|G\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right).$$

We estimate

$$\begin{aligned} \|\mathbf{T}\|_{D^{-1,\widehat{p}(\cdot)}(\mathbb{R}_>^n)} \\ = \sup_{\substack{\Psi \in C_0^\infty(\mathbb{R}_>^n), \\ \|\nabla \Psi\|_{\widehat{p}'(\cdot), \mathbb{R}_>^n} \leq 1}} \int_{\widehat{V}} \left(\widehat{\tau} \widehat{\mathbf{f}} \cdot \Psi - A_i \partial_i \widehat{\mathbf{v}} \cdot \partial_n \Psi - \widehat{\mathbf{v}} \cdot \partial_i (B_i \Psi) \right. \\ \left. + C \widehat{\mathbf{v}} \cdot \Psi - \widehat{\pi} D_j \partial_n \Psi_j + \widehat{\pi} \Psi_j E_j \right) dx \\ \leq \sup_{\substack{\Psi \in C_0^\infty(\mathbb{R}_>^n), \\ \|\nabla \Psi\|_{\widehat{p}'(\cdot), \mathbb{R}_>^n} \leq 1}} \int \mathbf{f} \cdot (\Psi \circ \mathbf{F}\tau) + \pi (\Psi_j E_j) \circ \mathbf{F} dx + c \|\widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \\ + \frac{1}{2} \left(\|\nabla \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right) \\ \leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(V)} + \|\pi\|_{W^{-1,p(\cdot)}(V)} + \|\widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right) \\ + \frac{1}{2} \left(\|\nabla \widehat{\mathbf{v}}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} + \|\widehat{\pi}\|_{L^{\widehat{p}(\cdot)}(\widehat{V})} \right) \end{aligned}$$

The equality is derived by integrating by parts the second, third and fifth summand. Note that the functions A_i and D_j do not depend on the last variable. For the first inequality we used Hölder's inequality, estimate (2.3) and again the fact that A_i and D_j can be made arbitrarily small by reducing α . To derive the second inequality note that $\Psi \circ \mathbf{F}\tau$ and $(\Psi_j E_j) \circ \mathbf{F}$ are in $W_0^{1,p'(\cdot)}(V)$ and that by estimate (2.3)

$$\|\Psi \circ \mathbf{F}\tau\|_{W^{1,p'(\cdot)}(V)} + \|(\Psi_j E_j) \circ \mathbf{F}\|_{W^{1,p'(\cdot)}(V)} \leq c \|\nabla \Psi\|_{L^{\widehat{p}'(\cdot)}(\mathbb{R}_>^n)}.$$

Furthermore we have

$$\|G\|_{\widehat{p}(\cdot), \widehat{V}} \leq \|\widehat{g}\|_{\widehat{p}(\cdot), \widehat{V}} + c\|\widehat{\mathbf{v}}\|_{\widehat{p}(\cdot), \widehat{V}} + \frac{1}{2}\|\nabla \widehat{\mathbf{v}}\|_{\widehat{p}(\cdot), \widehat{V}}.$$

Now, proceeding as in the derivation of (3.12)₂ we get (3.12)₁. \square

Now we can finally prove the main assertions of this paper.

Proof. (of the Theorems 20 and 21) Due to [Gal94, Theorem IV.6.1] for $\mathbf{f} \in C_0^\infty(\Omega)$ and $g \in C^\infty(\overline{\Omega})$ with $\int_\Omega g = 0$ there is a unique strong solution $(\mathbf{v}, \pi) \in W^{2,p(\cdot)}(\Omega) \times W^{1,p(\cdot)}(\Omega)$ of the Stokes system (3.1) with data \mathbf{f} , g and homogenous boundary condition. For each boundary point $\bar{x} \in \partial\Omega$ we may choose sets V and V' like in Theorem 29 and analogously defined sets Λ' and V'_- . Then, the sets $W' := V' \cup \Lambda' \cup V'_-$ form an open cover of the boundary. Since $\partial\Omega$ is compact, we may choose a finite subcover $W_i, i = 1, \dots, m$. Finally we may choose open sets $\Omega_0, \Omega_1 \subset \Omega$ such that $\Omega_0 \subset \subset \Omega_1 \subset \subset \Omega$ and $\Omega = \Omega_0 \cup \bigcup_{i=1}^m V'_i$. Now, using the Theorems 25 and 29 we conclude that

$$\begin{aligned} & \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} \\ & \leq \left\| \nabla \mathbf{v} \left(\chi_{\Omega_0} + \sum_i \chi_{V'_i} \right) \right\|_{L^{p(\cdot)}(\Omega)} + \left\| \pi \left(\chi_{\Omega_0} + \sum_i \chi_{V'_i} \right) \right\|_{L^{p(\cdot)}(\Omega)} \\ & \leq \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(\Omega_0)} + \sum_i \|\nabla \mathbf{v}\|_{L^{p(\cdot)}(V'_i)} + \|\pi\|_{L^{p(\cdot)}(\Omega_0)} + \sum_i \|\pi\|_{L^{p(\cdot)}(V'_i)} \\ (3.14) \quad & \leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega_1)} + \|g\|_{L^{p(\cdot)}(\Omega_1)} + \|\mathbf{v}\|_{L^{p(\cdot)}(\Omega_1 \setminus \Omega_0)} + \|\pi\|_{W^{-1,p(\cdot)}(\Omega_1 \setminus \Omega_0)} \right. \\ & \quad \left. + \sum_i \|\mathbf{f}\|_{W^{-1,p(\cdot)}(V_i)} + \|g\|_{L^{p(\cdot)}(V_i)} + \|\mathbf{v}\|_{L^{p(\cdot)}(V_i)} + \|\pi\|_{W^{-1,p(\cdot)}(V_i)} \right) \\ & \leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)} + \|\mathbf{v}\|_{L^{p(\cdot)}(\Omega)} + \|\pi\|_{W^{-1,p(\cdot)}(V_i)} \right), \end{aligned}$$

The last two summands may be eliminated as the subsequent lemma shows. Hence we get the estimate

$$(3.15) \quad \|\mathbf{v}\|_{W^{1,p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} \leq c \left(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)} \right).$$

Analogously one shows

$$\begin{aligned} & \|\nabla^2 \mathbf{v}\|_{L^{p(\cdot)}(\Omega)} + \|\nabla \pi\|_{L^{p(\cdot)}(\Omega)} \\ & \leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{W^{1,p(\cdot)}(\Omega)} + \|\mathbf{v}\|_{W^{1,p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} \right). \end{aligned}$$

Using (3.15) we conclude that

$$(3.16) \quad \|\mathbf{v}\|_{W^{2,p(\cdot)}(\Omega)} + \|\pi\|_{W^{1,p(\cdot)}(\Omega)} \leq c \left(\|\mathbf{f}\|_{L^{p(\cdot)}(\Omega)} + \|g\|_{W^{1,p(\cdot)}(\Omega)} \right).$$

Due to the estimates (3.15) and (3.16) we may continuously extend the linear solution operator to $L^{p(\cdot)}(\Omega) \times W^{1,p(\cdot)}(\Omega)$ and $W^{-1,p(\cdot)}(\Omega) \times L^{p(\cdot)}(\Omega)$, respectively. It is easy to see that these extensions map to strong and weak solutions of the Stokes system (3.1) with homogenous boundary conditions, respectively. Uniqueness is implied by $W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p^-}(\Omega)$ and [Gal94, Theorem IV.6.1]. Hence the subsequent lemma finishes the proof. \square

Lemma 30. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $(\mathbf{v}, \pi) \in (W^{2,p(\cdot)}(\Omega))^n \times W^{1,p(\cdot)}(\Omega)$ a strong solution of the Stokes system (3.1) with data $\mathbf{f} \in L^{p(\cdot)}(\Omega)$, $g \in W^{1,p(\cdot)}(\Omega)$ and homogenous boundary condition. Then we have the estimate*

$$\|\mathbf{v}\|_{L^{p(\cdot)}(\Omega)} + \|\pi\|_{W^{-1,p(\cdot)}(\Omega)} \leq c(\|\mathbf{f}\|_{W^{-1,p(\cdot)}(\Omega)} + \|g\|_{L^{p(\cdot)}(\Omega)})$$

with a constant $c = c(p, \Omega)$.

Proof. Let us assume that the estimate is wrong. This means that we have a sequence of solutions (\mathbf{v}_k, π_k) of the system with data \mathbf{f}_k and g_k which satisfy $\|\mathbf{v}_k\|_{L^{p(\cdot)}(\Omega)} + \|\pi_k\|_{W^{-1,p(\cdot)}(\Omega)} = 1$ and $\mathbf{f}_k \rightarrow 0$ in $W^{-1,p(\cdot)}(\Omega)$, $g_k \rightarrow 0$ in $L^{p(\cdot)}(\Omega)$. Since estimate (3.14) holds for (\mathbf{v}_k, π_k) with \mathbf{f}, g replaced by \mathbf{f}_k, g_k , we conclude, that (\mathbf{v}_k, π_k) is bounded in $W_0^{1,p(\cdot)}(\Omega) \times L^{p(\cdot)}(\Omega)$. Hence we find subsequences (again denoted by index k) satisfying

$$\begin{aligned} \mathbf{v}_k &\rightharpoonup \mathbf{v} \text{ in } W_0^{1,p(\cdot)}(\Omega), \quad \mathbf{v}_k \rightarrow \mathbf{v} \text{ in } L^{p(\cdot)}(\Omega), \\ \pi_k &\rightharpoonup \pi \text{ in } L^{p(\cdot)}(\Omega), \quad \pi_k \rightarrow \pi \text{ in } W^{-1,p(\cdot)}(\Omega). \end{aligned}$$

The strong convergences follow from the compact embeddings

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega),$$

and

$$L^{p(\cdot)}(\Omega) \hookrightarrow W^{-1,p(\cdot)}(\Omega).$$

Thus on one hand we have $\|\mathbf{v}\|_{L^{p(\cdot)}(\Omega)} + \|\pi\|_{W^{-1,p(\cdot)}(\Omega)} = 1$, and on the other hand for all $\phi \in (C_0^\infty(\Omega))^n$, $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} \int_{\Omega} \nabla \mathbf{v} \cdot \nabla \phi - \pi \operatorname{div} \phi \, dx &= \lim_k \int_{\Omega} \nabla \mathbf{v}_k \cdot \nabla \phi - \pi_k \operatorname{div} \phi \, dx = \lim_k \int_{\Omega} \mathbf{f}_k \cdot \phi \, dx = 0, \\ \int_{\Omega} \operatorname{div} \mathbf{v} \phi \, dx &= \lim_k \int_{\Omega} \operatorname{div} \mathbf{v}_k \phi \, dx = \lim_k \int_{\Omega} g_k \phi \, dx = 0. \end{aligned}$$

Hence (\mathbf{v}, π) is a weak solution of the system with zero data. From [Gal94, Theorem IV.6.1] we conclude that $\mathbf{v} \equiv 0$, $\pi \equiv 0$; a contradiction. \square

4. POISSON PROBLEM

Let us state in this final section the most important of the analogous results for the *Poisson problem*, cf. [DHHR10] for a sketch of the proof and [Len08] for full details. Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, with $C^{1,1}$ -boundary. Using the techniques¹ we employed in the case of Stokes system one can show that the Poisson problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= u_0 \text{ on } \partial\Omega, \end{aligned} \tag{4.1}$$

possesses unique strong and weak solutions, respectively, provided that the data have the appropriate regularity. More precisely, one can prove:

Theorem 31. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. For arbitrary data $f \in L^{p(\cdot)}(\Omega)$ and $u_0 \in \operatorname{tr}(W^{2,p(\cdot)}(\Omega))$ there exists a unique strong solution $u \in W^{2,p(\cdot)}(\Omega)$ of the Poisson equation (4.1) which satisfies the estimate*

$$\|u\|_{W^{2,p(\cdot)}(\Omega)} \leq c \left(\|f\|_{L^{p(\cdot)}(\Omega)} + \|u_0\|_{\operatorname{tr}(W^{2,p(\cdot)}(\Omega))} \right),$$

where the constant c depends only on the domain Ω and the exponent p .

Theorem 32. *Let $p \in \mathcal{P}^{\log}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. For arbitrary data $f \in W^{-1,p(\cdot)}(\Omega)$ and $u_0 \in \operatorname{tr}(W^{1,p(\cdot)}(\Omega))$ there exists a unique weak solution $u \in W^{1,p(\cdot)}(\Omega)$ of the Poisson equation (4.1) which satisfies the estimate*

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} \leq c' \left(\|f\|_{W^{-1,p(\cdot)}(\Omega)} + \|u_0\|_{\operatorname{tr}(W^{1,p(\cdot)}(\Omega))} \right),$$

where the constant c' depends only on the domain Ω and the exponent p .

¹As we already pointed out it is considerably simpler to obtain the half-space results in the case of the Poisson problem.

We call u a *strong solution* of (4.1) provided that it satisfies the differential equation in (4.1) in the sense of weak derivatives. We call u a *weak solution* of (4.1) provided that

$$\int_{\Omega} \nabla u \cdot \nabla \phi \, dx = \langle f, \phi \rangle \quad \forall \phi \in W_0^{1,p'(\cdot)}(\Omega),$$

$$u = u_0 \quad \text{on } \partial\Omega.$$

In order to show the above theorems one needs the following results which are also of interest on their own. Solutions of the equation

$$(4.2) \quad -\Delta u = f \text{ in } \mathbb{R}^n$$

are obtained by a convolution of f with the *Newton potential*

$$K(x) := \frac{1}{(n-2)|\partial B_1|} \frac{1}{|x|^{n-2}}.$$

It is well known and easy to see that the second derivatives of the Newton potential satisfy the assumptions of Theorem 18. Consequently we get:

Lemma 33. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$, and let $f \in C_{0,0}^\infty(\mathbb{R}^n)$. Then the convolution $u := K * f$ is infinitely differentiable and solves the problem (4.2). Moreover, the first and second order derivatives have the representations $(i, j = 1, \dots, n)$*

$$\partial_i u(x) = \int_{\mathbb{R}^n} \partial_{x_i} K(x-y) f(y) \, dy,$$

$$\partial_i \partial_j u(x) = \lim_{\epsilon \searrow 0} \int_{(B(x,\epsilon))^c} \partial_{x_i} \partial_{x_j} K(x-y) f(y) \, dy - \frac{1}{n} \delta_{ij} f(x),$$

and satisfy the estimates

$$\|\nabla u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)},$$

$$\|\nabla^2 u\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

with a constant $c = c(p)$.

Letting L denote the continuation of the operator $f \mapsto K * f$ in the appropriate spaces we can state the following theorem.

Theorem 34. *Let $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ satisfy $1 < p^- \leq p^+ < \infty$.*

(1) *If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ then $Lf \in D^{2,p(\cdot)}(\mathbb{R}^n)$ satisfies the estimate*

$$\|Lf\|_{D^{2,p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

with a constant $c = c(p)$.

(2) *If $f \in D^{-1,p(\cdot)}(\mathbb{R}^n)$ then $Lf \in D^{1,p(\cdot)}(\mathbb{R}^n)$ satisfies the estimate*

$$\|Lf\|_{D^{1,p(\cdot)}(\mathbb{R}^n)} \leq c \|f\|_{D^{-1,p(\cdot)}(\mathbb{R}^n)}$$

with a constant $c = c(p)$.

(3) *If $f \in L^{p(\cdot)}(\mathbb{R}^n)$ has bounded support and vanishing mean value, hence $f \in D^{-1,p(\cdot)}(\mathbb{R}^n)$, then $Lf \in (D^{(1,2),p(\cdot)}(\mathbb{R}^n))^n \subset (D^{1,p(\cdot)}(\mathbb{R}^n))^n$ satisfies both of the above estimates.*

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